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Linear Sum of Two Subspaces :→ Let, U and W be two subspaces of a vector space V over a field F. Then the subset $\{u+w : u\in U, w\in W\}$ is said to be the linear sum of the subspaces U and W.

Let, $\alpha\in U$. Then $\alpha=\alpha+\theta$, where $\alpha\in U$ and $\theta\in W$. This shows that $\alpha\in U+W$. Therefore $U\subset U+W$.
Let, $\alpha\in W$. Then $\alpha=\theta+\alpha$, where $\theta\in U$, and $\alpha\in W$. This shows that $\alpha\in U+W$. Therefore $W\subset U+W$.

Theorem :→ Let, U and W be two subspaces of a vector space V over a field F. Then the linear sum $U+W$ is a vector subspace of V.

Proof :→ Let, $S=U+W=\{u+w : u\in U, w\in W\}$.
Now $\theta\in U$, $\theta\in W \Rightarrow \theta+\theta=\theta\in S$ and therefore S is non-empty.
Let, $\alpha_1,\alpha_2\in S$. Then $\alpha_1=u_1+w_1$, for some $u_1\in U$ & $w_1\in W$.
and $\alpha_2=u_2+w_2$, for some $u_2\in U$ & $w_2\in W$
Now $\alpha_1+\alpha_2=(u_1+w_1)+(u_2+w_2)=(u_1+u_2)+(w_1+w_2)\in S$, since $u_1+u_2\in U$ and $w_1+w_2\in W$.
Let, c be a scalar in F.
Then $c\alpha_1=c(u_1+w_1)=cu_1+cw_1\in S$, since $cu_1\in U$ and $cw_1\in W$.
Therefore $\alpha_1\in S, \alpha_2\in S \Rightarrow \alpha_1+\alpha_2\in S$ and $c\in F, \alpha_1\in S \Rightarrow c\alpha_1\in S$.
This proves that S is a subspace of V, i.e. $U+W$ is a subspace of V.

Theorem :→ The subspace $U+W$ is the smallest subspace of V containing the subspaces U and W.

Proof :→ Let, P be any subspace of V containing the subspaces U and W.
Now let, α be an element of $U+W$.
Then $\alpha=u_1+w_1$, for some $u_1\in U$, $w_1\in W$.
Since $U\subset P$, $u_1\in P$ and since $W\subset P$, $w_1\in P$.
Now since P is a subspace of V, and $u_1, w_1\in P$, so $u_1+w_1\in P$ i.e. $\alpha\in P$.

Thus $\alpha \in U+W \Rightarrow \alpha \in P$, and therefore $U+W \subset P$.
This proves that $U+W$ is the smallest subspace containing U and V.

Linear Combination of vectors :→ Let, V be a vector space over a field F. Let, $\alpha_1, \alpha_2, \ldots, \alpha_r \in V$. A vector β in V is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$ if β can be expressed as
$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r$ for some scalars $c_1, c_2, \ldots, c_r$ in F.

Example :→ Let, V be a real vector space and $\alpha, \beta, \gamma \in V$. Then $\alpha+\beta+\gamma$, $\alpha+2\beta+3\gamma$, $\alpha = 1\alpha + 0\beta + 0\cdot\gamma$, $\beta = 0\cdot\alpha + 1\cdot\beta + 0\cdot\gamma$, $\gamma = 0\cdot\alpha + 0\cdot\beta + 1\cdot\gamma$ are linear combinations of α, β, γ.

Theorem :→ Let, V be a vector space over a field F and let, S be a non-empty finite subset of V. Then the set W of all linear combinations of the vectors in S forms a subspace of V and this is the smallest subspace containing the set S.

Proof :→ Let, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then $W = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n : c_1, c_2, \ldots, c_n \in F\}$.
W is a non-empty subset of V, since $\alpha_1, \alpha_2, \ldots, \alpha_n \in W$.
Let, $\alpha = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n \in W$, $\beta = s_1\alpha_1 + s_2\alpha_2 + \cdots + s_n\alpha_n \in W$.
Then $r_1, r_2, \ldots, r_n \in F$ and $s_1, s_2, \ldots, s_n \in F$.
$\therefore \alpha+\beta = (r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n) + (s_1\alpha_1 + s_2\alpha_2 + \cdots + s_n\alpha_n)$.
$= (r_1+s_1)\alpha_1 + (r_2+s_2)\alpha_2 + \cdots + (r_n+s_n)\alpha_n \in W$, since $r_i + s_i \in F$, for $1 \le i \le n$. So $\alpha \in W$ and $\beta \in W \Rightarrow \alpha+\beta \in W$ ——→ (1)
Let, $p \in F$. Then $p\alpha = p(r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n)$
$= (pr_1)\alpha_1 + (pr_2)\alpha_2 + \cdots + (pr_n)\alpha_n \in W$,
since $pr_i \in F$, for $1 \le i \le n$. So $p \in F, \alpha \in W \Rightarrow p\alpha \in W$ ——→ (2)
From (1) and (2) it follows that W is a subspace of V.
Let, P be any subspace of V containing the set S.
Let, $\gamma \in W$. Then $\gamma = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$ for some scalars $x_i \in F$ for $1 \le i \le n$.
Since P is a subspace of V containing α_i and $x_i \in F$, then $x_i\alpha_i \in P$ for $1 \le i \le n$. Since P is a subspace and $x_1\alpha_1, x_2\alpha_2, \ldots, x_n\alpha_n \in P$, so $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n \in P$, i.e. $\gamma \in P$.
Thus $\gamma \in W \Rightarrow \gamma \in P$ and therefore $W \subset P$.
This proves that W is the smallest subspace containing S.